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Theorem Let X and Y be two metric spaces and f be a mapping of X into Y. Then f is continuous at x_0 iff

$$x_n \to x_0 \Rightarrow f(x_n) \to f(x_0).$$

Proof Necessary part

Let f is continuous at x_0.

Let $\{x_n\}$ be a sequence in X such that $x_n \to x_0$.

We have to prove that $f(x_n) \to f(x_0)$.

Let $S_\epsilon(f(x_0))$ be an open sphere with centre on $f(x_0)$.

Since f is continuous at x_0, by definition there exists an open sphere $S_\delta(x_0)$ centred on x_0 such that $f(S_\delta(x_0)) \subseteq S_\epsilon(f(x_0))$. —(1)

$\because$ $x_n \to x_0$

$\Rightarrow$ all x_n's from some place on lie in $S_\epsilon(f(x_0))$.

Since $f(S_\delta(x_0)) \subseteq S_\epsilon(f(x_0))$ [using (1)]

$\Rightarrow$ all $f(x_n)$'s from some place on lie in $S_\epsilon(f(x_0))$.

$\Rightarrow f(x_n) \longrightarrow f(x_0)$

[Necessary part proved]

Sufficient part

Given that $x_n \to x_0 \Rightarrow f(x_n) \to f(x_0)$.

we have to prove that f is continuous at x_0.

Let us suppose that f is not continuous at x_0.

We shall show that

$x_n \to x_0$ does not imply $f(x_n) \to f(x_0)$.

Since f is not continuous, $\exists$ an open sphere $S_\epsilon(f(x_0))$ with the property that image under f of each open sphere with centre at x_0 is not contained in it.

we consider the sequence of open spheres $S_1(x_0), S_{1/2}(x_0), \ldots, S_{1/n}(x_0), \ldots$. we form a sequence $\{x_n\}$ such that $x_n \in S_{1/n}(x_0)$ and $f(x) \notin S_\epsilon(f(x_0))$.

$\Rightarrow$ x_n converges to x_0 but $f(x_n)$ doesnot converge to $f(x_0)$.

$\Rightarrow$ our suppositions are wrong.

$\Rightarrow$ f is continuous at x_0.